

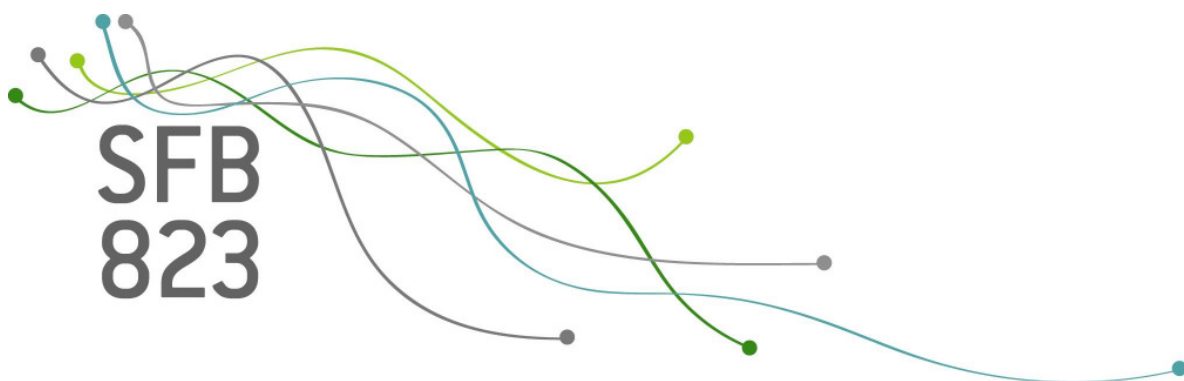
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# Some comments on goodness-of-fit tests for the parametric form of the copula based on $L^2$ -distances

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Discussion Paper





# Some comments on goodness-of-fit tests for the parametric form of the copula based on $L^2$ -distances

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## Abstract

In a recent paper Fermanian (2005) studied a goodness-of-fit test for the parametric form of a copula, which is based on an  $L^2$ -distance between a parametric and nonparametric estimate of the copula density. In the present paper we investigate the asymptotic properties of the proposed test statistic under fixed alternatives. We also study the impact of different estimates for the parameters of the finite dimensional family of copulas specified by the null hypothesis and illustrate the performance of a parametric bootstrap procedure for the approximation of the critical values.

Keywords and Phrases: goodness-of-fit tests, Copula,  $L^2$ -distance, parametric bootstrap

AMS Subject Classification: 62G; 62P

## 1 Introduction

Nowadays copulas are widely used by practitioners to analyze dependence structures in various applications including finance, actuarial science and hydrology [see e.g. Frees and Valdez (1998), Embrechts, McNeil and Straumann (2002), McNeil, Frey and Embrechts (2005) and Genest and Favre (2007)]. The copula of a multivariate distribution describes its dependence structure as a complement to the behaviour of its margins, and as a consequence the estimation of the distribution can be splitted into the estimation of the marginals and the copula. Several parametric families of copulas

$$(1.1) \quad \mathcal{C} = \{C_\theta \mid \theta \in \Theta\}$$

(here  $\Theta \subset \mathbb{R}^q$  denotes an arbitrary subset) have been proposed in the literature in order to reflect various aspects of dependency [see e.g. the monographs of Joe (1997) or Nelsen (2006)]. Because misspecification of the copula function can have a serious impact on the statistical analysis, several authors have pointed out the importance of goodness-of-fit tests for the hypothesis of a parametric family (1.1) [see e.g. Malevergne and Sornette (2003), Cui and Sun (2004), Fermanian (2005), Scaillet (2007), Dobric and Schmid (2007), Genest and Rémillard (2008) among many others]. In a recent paper Fermanian (2005) proposed a test for the hypothesis

$$(1.2) \quad H_0 : C \in \mathcal{C}, \quad H_1 : C \notin \mathcal{C}$$

where  $C$  denotes the copula of a  $d$ -dimensional distribution and  $\mathcal{C}$  the parametric class defined by (1.1). The test is based on an  $L^2$ -distance between a nonparametric and parametric estimate of the copula density, and Fermanian (2005) proved asymptotic normality of the corresponding test statistic under the null hypothesis  $H_0$  and specific assumptions on the estimates of the parameters of the copula density.

The present paper has several purposes. First we provide a more sophisticated analysis of the test proposed by Fermanian (2005) and study the asymptotic properties of the test statistic under fixed alternatives. It is shown that in this case an appropriately standardized version of the test statistic is also asymptotically normal distributed. Moreover, in contrast to the null hypothesis, it turns out that under the alternative the form of the parametric estimate has a substantial impact on the asymptotic distribution of the test statistic. In particular, we investigate two estimation methods for the parameter of the copula density, namely the common maximum likelihood and minimum  $L^2$ -distance estimation technique. For these estimates we prove asymptotic normality of the standardized test statistic under the null hypothesis and fixed alternatives with different rates of convergence in both cases. These results can be used for the construction of confidence regions for a measure of deviation, say  $M^2$ , between the parametric family of copulas and the "true" copula or for testing *precise hypotheses* of the form  $H_0 : M^2 \leq \Delta$  vs.  $H_1 : M^2 > \Delta$ , where  $\Delta \geq 0$  is a preassigned measure of accuracy [see Berger and Delampady (1987)]. These hypotheses are motivated by the observation that in practice a copula will never be exactly of a given parametric form (which would correspond to the case  $\Delta = 0$ ), but in the best case approximately given by a parametric form (which would correspond to a small value of  $\Delta$ ).

Secondly it has been pointed out by several authors that under the null hypothesis the normal approximation of test statistics based on  $L^2$ -distances is not very accurate [see e.g. Härdle and Mammen (1993) or Fan and Linton (2003)]. Therefore we propose a parametric bootstrap procedure for the approximation of the critical values and investigate its performance by means of a simulation study. In particular, it is demonstrated that the bootstrap test based on the  $L^2$ -distance yields a reliable approximation of the nominal level and has similar power properties as several goodness-of-fit tests, which were investigated in a recent paper by Genest, Rémillard and Beaudoin (2008).

The remaining part of the paper is organized as follows. In Section 2 we introduce the necessary notation and define the test statistic. The asymptotic properties of the maximum likelihood and the minimum  $L^2$ -distance in the parametric family of copulas are investigated in Section 3 under

a correct and incorrect specification of the parametric family of copulas. In Section 4 we prove asymptotic normality of the test statistic under the null hypothesis and fixed alternatives with different rates of convergence in both cases. Section 5 is devoted to a small simulation study in order to investigate the finite sample properties of a parametric bootstrap procedure based on the  $L^2$ -distance. Finally, some more technical results are presented in the Appendix.

## 2 Testing for the form of the copula with the $L^2$ -distance

Throughout this paper let  $X_1, \dots, X_n$  denote independent identically distributed  $d$ -dimensional random variables with joint continuous distribution function  $H$  and copula  $C$ , which has a density  $\tau_0$  supported in the cube  $[0, 1]^d$ . We denote by  $X_i = (X_{i,1}, \dots, X_{i,d})^T$  the components of the vector  $X_i$  ( $i = 1, \dots, n$ ) and by  $F_j$  the marginal distribution of the  $j$ th component ( $j = 1, \dots, d$ ), which yields by Sklar's theorem [see e.g. Nelsen (2006)]

$$(2.1) \quad H(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)) \quad \text{a.e. .}$$

We define by

$$(2.2) \quad F_{n,r}(x) = \frac{1}{n} \sum_{j=1}^n \mathbb{1}\{X_{j,r} \leq x\}$$

the empirical distribution function of the  $r$ -th marginal distribution  $F_r$  ( $r = 1, \dots, d$ ) and

$$(2.3) \quad Y_i := (F_1(X_{i,1}), \dots, F_d(X_{i,d}))^T$$

$$(2.4) \quad Y_{n,i} := (F_{n,1}(X_{i,1}), \dots, F_{n,d}(X_{i,d}))^T,$$

(note that the distribution function of the random variable  $Y_i$  is given by the copula  $C$ ). We assume for the parametric class of copulas defined by (1.1) that the parameter space  $\Theta$  is compact with non empty interior and that  $C_\theta$  has a density supported in  $[0, 1]^d$ , say  $\tau(\cdot, \theta)$ , which is two and three times continuously differentiable with respect to the first and second argument, respectively. In the following discussion  $\hat{\theta}$  denotes an (under the null hypothesis consistent) estimate of the parameter  $\theta$ , which will be specified in the following section, and we denote by  $\hat{\tau}(\cdot) = \tau(\cdot, \hat{\theta})$  the corresponding parametric estimate of the copula density. Moreover, the nonparametric kernel estimate of the copula density is defined by

$$(2.5) \quad \tau_n(u) := \frac{1}{nh^d} \sum_{i=1}^n K\left(\frac{u - Y_{n,i}}{h}\right)$$

[see e.g. Charpentier, Fermanian and Scaillet (2007)] and  $\omega : [0, 1]^d \longrightarrow \mathbb{R}_0^+$  denotes a two times continuously differentiable weight function with compact support contained in the cube  $[\varepsilon_1, 1 - \varepsilon_1]^d \subset [0, 1]^d$ , where  $\varepsilon_1 > 0$ . We further assume that  $\tau(\cdot, \cdot)$  is uniformly continuous on

$[\varepsilon_0, 1 - \varepsilon_0]^d \times \Theta$  for some  $\varepsilon_0 \in (0, \varepsilon_1)$ . For testing the parametric hypothesis (1.2) Fermanian (2005) proposed the test statistic

$$(2.6) \quad J_n = J_n(\hat{\theta}) := \int (\tau_n - K_h * \hat{\tau})^2(u) \omega(u) du.$$

where  $*$  denotes convolution and  $K_h(x) = K(x/h)/h$ . Here  $K$  denotes the kernel of the density estimate (2.5), and the convolution operator is applied to the parametric estimate of the copula density in order to reduce the bias [see e.g. Bickel and Rosenblatt (1973) or Härdle and Mammen (1993)]. Because the asymptotic properties of the test statistic, especially under a fixed alternative, depend sensitively on the specific choice of the parametric estimate, we study in the following section two estimation methods for the parameter  $\theta$  of the family of copula densities.

### 3 Estimation of the parameter of the copula density

Throughout this paper let  $\tau_0$  denote the “true” density of the copula  $C$  and consider the Kulback Leibler distance

$$(3.1) \quad D_{KL}(\tau_0, \tau(\cdot, \theta)) = \int \log(\tau_0(u))\tau_0(u) du - \int \log(\tau(u, \theta))\tau_0(u) du,$$

We define  $L(\theta) = \int \log \tau(u, \theta)\tau_0(u) du$  and

$$(3.2) \quad \theta_{ML}^* = \operatorname{argmax}_{\theta \in \Theta} L(\theta)$$

as the parameter corresponding to the best approximation of the copula density  $\tau_0$  by the parametric class  $\{\tau(\cdot, \theta) \mid \theta \in \Theta\}$  with respect to the Kulback Leibler distance and assume that  $\theta_{ML}^*$  is attained at an unique interior point of  $\Theta$ . The maximum likelihood estimate of the parameter  $\theta$  is defined as

$$(3.3) \quad \hat{\theta}_{ML} := \operatorname{argmax}_{\theta \in \Theta} L_n(\theta).$$

where  $L_n(\theta) := \frac{1}{n} \sum_{i=1}^n \log \tau(Y_{n,i}, \theta)$  denotes the likelihood function [see Genest, Ghouli and Rivest (1995)]. We assume that  $\hat{\theta}_{ML}$  is also attained at an interior point of the parameter space  $\Theta$  and that the parametric class of copula densities satisfies the following assumptions of regularity:

- (a)  $\mathbb{E} [\|\partial_\theta \log \tau(Y_i, \theta_{ML}^*)\| + \|\partial_{y\theta}^2 \log \tau(Y_i, \theta_{ML}^*)\| + \|\partial_{yy\theta}^3 \log \tau(Y_i, \theta_{ML}^*)\|] < \infty$
- (b) There exist constants  $\alpha, \beta > 0$ , such that for any point  $Y_{ni}^*$  with  $\|Y_{ni}^* - Y_i\| \leq \|Y_{ni} - Y_i\|$ 

$$\|\partial_{yy\theta}^3 \log(Y_{ni}^*, \theta_{ML}^*)\| \leq \alpha \|\partial_{yy\theta}^3 \log \tau(Y_i, \theta_{ML}^*)\| + \beta \|\partial_{yy\theta}^3 \log \tau(Y_{ni}, \theta_{ML}^*)\|$$
- (c) For all  $u \in (0, 1)^d$  we have with the notation  $r(t) := t(1 - t)$

$$\|\partial_{yy\theta}^3 \log \tau(u, \theta_{ML}^*)\| \leq \text{const } r(u_1)^{a_1} \dots r(u_d)^{a_d},$$

where  $a_k = \frac{\delta-1}{p_k}$  mit  $\frac{1}{p_1} + \dots + \frac{1}{p_d} = 1$  and  $\delta > 0$ .

(d) For all  $u \in (0, 1)^d$

$$\|\partial_{\theta\theta}^2 \log \tau(u, \theta_{ML}^*)\| \leq \text{const } r(u_1)^{b_1} \dots r(u_d)^{b_d},$$

where  $b_k = \frac{\zeta-1}{q_k}$  and  $\frac{1}{q_1} + \dots + \frac{1}{q_d} = 1$ ,  $\zeta > 0$ .

(e) For all  $u \in (0, 1)^d$  we have in a neighbourhood  $V(\theta_{ML}^*)$  of the point  $\theta_{ML}^*$

$$\sup_{\theta \in V(\theta_{ML}^*)} \|\partial_{\theta\theta\theta}^3 \log \tau(u, \theta)\| \leq \text{const } r(u_1)^{c_1} \dots r(u_d)^{c_d},$$

where  $c_k = \frac{\eta-1}{p'_k}$  and  $\frac{1}{p'_1} + \dots + \frac{1}{p'_d} = 1$ ,  $\eta > 0$ .

Fermanian (2005) states that these assumptions are satisfied for most of the commonly used copula families [see also Hu (1998), who proved some of these assumptions for Clayton-, Frank- and Gauß-copulas]. Our first result establishes a stochastic expansion for the maximum likelihood estimate, from which asymptotic normality can be derived.

**Theorem 3.1.** *If the assumptions stated in Section 2 and 3 are satisfied and the maximum likelihood estimate  $\hat{\theta}_{ML}$  is consistent, i.e.  $\hat{\theta}_{ML} \xrightarrow{P} \theta_{ML}^*$ , then*

$$(3.4) \quad \sqrt{n}(\hat{\theta}_{ML} - \theta_{ML}^*) = \frac{1}{\sqrt{n}} \sum_{i=1}^n D(Y_i) + o_P(1) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma),$$

where

$$\begin{aligned} D(Y_i) &= A^{-1}(\partial_{\theta} \log \tau(Y_i, \theta_{ML}^*) + h(Y_i)), \\ h(Y_i) &= \int \partial_{y\theta}^2 \log \tau(u, \theta_{ML}^*)(\mathbb{1}(Y_i \leq u) - u) \tau_0(u) du, \\ A &= -\mathbb{E} [\partial_{\theta}^2 \log \tau(Y_i, \theta_{ML}^*)], \\ \Sigma &= \text{Var}(D(Y_i)). \end{aligned}$$

**Proof.** The proof follows by similar arguments as presented in part D of the paper by Fermanian (2005). Because this author only considers the null hypothesis and some of the arguments in this reference seem to be incorrect (see our Remark 3.2), we present here the main steps. Because  $\hat{\theta}_{ML}$  is an interior point of the parameter space, we obtain by means of Taylor expansions

$$(3.5) \quad 0 = \partial_{\theta} L_n(\hat{\theta}_{ML}) = S_0 + S_1 + S_2 + \partial_{\theta\theta}^2 L_n(\theta_{ML}^*)(\hat{\theta}_{ML} - \theta_{ML}^*) + \frac{1}{2}(\hat{\theta}_{ML} - \theta_{ML}^*)^T \partial_{\theta\theta\theta}^3 L_n(\tilde{\theta})(\hat{\theta}_{ML} - \theta_{ML}^*),$$

where the terms  $S_0, S_1, S_2$  are defined by

$$(3.6) \quad S_0 = \frac{1}{n} \sum_{i=1}^n \partial_{\theta} \log \tau(Y_i, \theta_{ML}^*),$$

$$(3.7) \quad S_1 = \frac{1}{n} \sum_{i=1}^n \partial_{y\theta}^2 \log \tau(Y_i, \theta_{ML}^*)(Y_{n,i} - Y_i),$$

$$(3.8) \quad S_2 = \frac{1}{2n} \sum_{i=1}^n (Y_{n,i} - Y_i)^T \partial_{yy\theta}^3 \log \tau(Y_{n,i}^*, \theta_{ML}^*)(Y_{n,i} - Y_i),$$

and  $\tilde{\theta}$  and  $Y_{ni}^*$  satisfy  $\|\tilde{\theta} - \theta_{ML}^*\| \leq \|\hat{\theta}_{ML} - \hat{\theta}_{ML}^*\|$  and  $\|Y_{ni}^* - Y_i\| \leq \|Y_{n,i} - Y_i\|$  ( $i = 1, \dots, n$ ), respectively. Note that  $E[S_0] = 0$  and therefore  $S_0$  is a sum of centered, independent and identically distributed random variables. Using a Hoeffding approximation it follows for the second term

$$(3.9) \quad S_1 = \frac{1}{n} \sum_{i=1}^n h(Y_i) + o_P\left(\frac{1}{\sqrt{n}}\right),$$

where the random variable  $h(Y_i)$  is defined in Theorem 3.1. Finally, we have (using the assumptions (a) - (c)) that  $S_2 = O_P(\frac{\log_2 n}{n}) = o_P(\frac{1}{\sqrt{n}})$ . Combining these estimates we obtain

$$(3.10) \quad \partial_{\theta} L_n(\theta_{ML}^*) = S_0 + S_1 + S_2 = \frac{1}{n} \sum_{i=1}^n A \cdot D(Y_i) + o_P\left(\frac{1}{\sqrt{n}}\right),$$

where we have again used the notation of Theorem 3.1. Note that it follows from Proposition A.1 in Genest, Ghouli and Rivest (1995) that  $\partial_{\theta}^2 L_n(\theta_{ML}^*) \xrightarrow{P} -A$ , which implies (observing that  $\partial_{\theta\theta\theta}^3 L_n(\tilde{\theta})$  is bounded in probability, because of assumption (e) and  $\tilde{\theta} \xrightarrow{P} \theta_{ML}^*$ )

$$(3.11) \quad -\partial_{\theta} L_n(\theta_{ML}^*) = (-A + o_P(1))(\hat{\theta}_{ML} - \theta_{ML}^*)$$

The assertion of the Theorem now follows from (3.5) and (3.10) by a standard argument.  $\square$

### Remark 3.2.

- (a) Note that Theorem 3.1 requires the consistency of the maximum likelihood estimate  $\hat{\theta}_{ML} \xrightarrow{P} \theta_{ML}^*$ , where  $\theta_{ML}^*$  is the best approximation of the “true” copula density  $\tau_0$  by the parametric class  $\{\tau(\cdot, \theta) | \theta \in \Theta\}$  with respect to the Kulback Leibler distance. If the parametric model has been correctly specified, this was proved in Theorem 4.2.1 of Hu (1998) and similar arguments could be used to establish consistency of  $\hat{\theta}_{ML}$  if the model has been misspecified.
- (b) It should be pointed out here that Fermanian (2005) considered estimates of the parameter  $\theta$  satisfying

$$\hat{\theta} - \theta_0 = \frac{1}{n} A^{-1}(\theta_0) \sum_{i=1}^n B(\theta_0, Y_i) + o_p(n^{-1/2}(\log n)^{-1/2})$$

where  $\theta_0$  is the “true” parameter of the copula (this means that the model has been correctly specified). However, we were not able to find estimates in the literature satisfying this assumption (in particular the proof presented in Appendix D of Fermanian (2005) seems to be not correct).

In the remaining part of this section we derive a similar expansion for an estimate minimizing an  $L^2$ -distance between the “true” copula density and the parametric class of densities specified by the null hypothesis. To be precise we define

$$(3.12) \quad \theta_{L^2}^* := \operatorname{argmin}_{\theta \in \Theta} \int (\tau(u, \theta) - \tau_0(u))^2 \omega(u) du$$



as the parameter corresponding to the best  $L^2$ -approximation of  $\tau_0$  by the parametric class  $\{\tau(\cdot, \theta) | \theta \in \Theta\}$  and we assume that it is attained at an unique interior point of the parameter space  $\Theta$ . If the model is correctly specified and  $\theta_0$  the true parameter corresponding to the density  $\tau_0(\cdot) = \tau(\cdot, \theta_0)$ , then  $\theta_{L^2}^* = \theta_0$ . The empirical analogue of (3.12) is given by

$$(3.13) \quad \hat{\theta}_{L^2} = \operatorname{argmin}_{\theta \in \Theta} \int (\tau(u, \theta) - \tau_n(u))^2 \omega(u) du,$$

where  $\tau_n$  denotes the kernel estimate of the copula defined by (2.5). The following result provides a stochastic expansion for the difference  $\hat{\theta}_{L^2} - \theta_{L^2}^*$ .

**Theorem 3.3.** *If the  $L^2$ -estimate is consistent, i.e.  $\hat{\theta}_{L^2} \xrightarrow{P} \theta_{L^2}^*$ , and the conditions  $h \rightarrow 0$ ,  $nh^d \rightarrow \infty$ ,*

$$(3.14) \quad \frac{\log_2^2 n}{nh^{4+\frac{d}{2}}} \rightarrow 0$$

$$(3.15) \quad \frac{\log(h^{-d})}{nh^d} \rightarrow 0$$

$$(3.16) \quad \frac{\log(h^{-d})}{\log_2 n} \rightarrow \infty$$

$$(3.17) \quad n^{1-\alpha} h^d \rightarrow \infty$$

are satisfied for some  $\alpha \in (0, \frac{3}{4})$ , then it follows that

$$(3.18) \quad \hat{\theta}_{L^2} - \theta_{L^2}^* + B = \frac{1}{n} \sum_{i=1}^n D_n(Y_i) + o_P\left(\frac{1}{\sqrt{n}}\right),$$

where the bias is given by

$$(3.19) \quad B := \int (K_h * \tau_0(u) - \tau_0(u)) C^{-1} \partial_\theta \tau(u, \theta_{L^2}^*) \omega(u) du = O(h^2).$$

and

$$(3.20) \quad \delta(u) := \tau(u, \theta_{L^2}^*) - \tau_0(u)$$

$$(3.21) \quad C = \int \{ \partial_\theta \tau(u, \theta_{L^2}^*) \partial_\theta^T \tau(u, \theta_{L^2}^*) + \delta(u) \partial_{\theta\theta}^2 \tau(u, \theta_{L^2}^*) \} \omega(u) du.$$

$$(3.22) \quad D_n(Y_i) := C^{-1} \left[ \int (K_h(u - Y_i) - \mathbb{E}[K_h(u - Y_i)]) \partial_\theta \tau(u, \theta_{L^2}^*) \omega(u) du + r_n(Y_i) \right]$$

$$(3.23) \quad r_n(Y_i) := \mathbb{E}[h_n(Y_k, Y_i) | Y_i]$$

$$(3.24) \quad h_n(Y_k, Y_i) := \frac{-1}{h} \int (dK)_h(u - Y_k) (\mathbb{1}(Y_i \leq Y_k) - Y_k) \partial_\theta \tau(u, \theta_{L^2}^*) \omega(u) du.$$

In particular, we have

$$(3.25) \quad \frac{1}{\sqrt{n}} \sum_{i=1}^n D_n(Y_i) \xrightarrow{\mathcal{D}} \mathcal{N}_q(0, \Sigma),$$

where the asymptotic variance is given by

$$\begin{aligned}
(3.26) \quad \Sigma = & C^{-1} \mathbb{E} \left[ \partial_{\theta} \tau(Y_i, \theta_{L^2}^*) \partial_{\theta}^T \tau(Y_i, \theta_{L^2}^*) \omega^2(Y_i) \right. \\
& - \mathbb{E} [\partial_{\theta} \tau(Y_i, \theta_{L^2}^*) \omega(Y_i)] \mathbb{E} [\partial_{\theta}^T \tau(Y_i, \theta_{L^2}^*) \omega(Y_i)] \\
& - 2 \sum_{r=1}^d \int \partial_{\theta} \tau(Y_i, \theta_{L^2}^*) \omega(Y_i) (\tau_0 \partial_{\theta}^T \tau(\cdot, \theta_{L^2}^*) \omega)(v_1, \dots, Y_{ir}, \dots, v_d) d\underline{v}_{-r} \\
& + \sum_{r,s=1}^d \int (\tau_0 \partial_{\theta}(\cdot, \theta_{L^2}^*) \omega)(u_1, \dots, Y_{ir}, \dots, u_d) (\tau_0 \partial_{\theta}^T(\cdot, \theta_{L^2}^*) \omega) \\
& \left. (v_1, \dots, Y_{is}, \dots, v_d) d\underline{u}_{-r} d\underline{v}_{-s} \right] C^{-1}
\end{aligned}$$

and the symbol  $d\underline{u}_{-r}$  means integration with respect to the  $(d-1)$ -dimensional variable  $\underline{u}_{-r} = (u_1, \dots, u_{r-1}, u_{r+1}, \dots, u_d)$ .

**Proof.** With the notation

$$\begin{aligned}
Q_n(\theta) &= - \int (\tau(u, \theta) - \tau_n(u))^2 \omega(u) du, \\
Q(\theta) &= - \int (\tau(u, \theta) - \tau_0(u))^2 \omega(u) du, \\
\psi_n(\theta) &= \partial_{\theta} Q_n(\theta) \\
\psi(\theta) &= \partial_{\theta} Q(\theta)
\end{aligned}$$

we obtain for same  $\tilde{\theta}$  with  $\|\tilde{\theta} - \theta_{L^2}^*\| \leq \|\hat{\theta}_{L^2} - \theta_{L^2}^*\|$  the expansion

$$\begin{aligned}
(3.27) \quad 0 &= \partial_{\theta} Q_n(\hat{\theta}_{L^2}) = \partial_{\theta} Q_n(\theta_{L^2}^*) \\
&+ \partial_{\theta}^2 Q_n(\theta_{L^2}^*) (\hat{\theta}_{L^2} - \theta_{L^2}^*) + \frac{1}{2} (\hat{\theta}_{L^2} - \theta_{L^2}^*)^T \partial_{\theta}^3 Q_n(\tilde{\theta}) (\hat{\theta}_{L^2} - \theta_{L^2}^*).
\end{aligned}$$

Moreover,

$$\begin{aligned}
(3.28) \quad \partial_{\theta} Q_n(\theta_{L^2}^*) &= \frac{2}{n} \sum_{i=1}^n \int (K_h(u - Y_{n,i}) - \tau(u, \theta_{L^2}^*)) \partial_{\theta} \tau(u, \theta_{L^2}^*) \omega(u) du \\
&= 2 \left( A_{n1} + A_{n2} + A_{n3} + A_{n4} \right),
\end{aligned}$$

where the random variables  $A_{nj}$  ( $j = 1, \dots, 4$ ) are defined by

$$\begin{aligned} A_{n1} &= \frac{1}{n} \sum_{i=1}^n \int (K_h(u - Y_i) - (K_h * \tau_0)(u)) \partial_\theta \tau(u, \theta_{L^2}^*) \omega(u) du, \\ A_{n2} &= \frac{1}{n} \sum_{i=1}^n \int \beta_{ni}(u) \partial_\theta \tau(u, \theta_{L^2}^*) \omega(u) du, \\ A_{n3} &= \frac{1}{n} \sum_{i=1}^n \int \gamma_{ni}^*(u) \partial_\theta \tau(u, \theta_{L^2}^*) \omega(u) du, \\ A_{n4} &= \int (K_h * \tau_0(u) - \tau(u, \theta_{L^2}^*)) \partial_\theta \tau(u, \theta_{L^2}^*) \omega(u) du, \end{aligned}$$

and

$$\begin{aligned} (3.29) \quad \alpha_i(u) &:= K_h(u - Y_i), \\ \beta_{ni}(u) &:= \frac{-1}{h} (dK)_h(u - Y_i) (Y_{n,i} - Y_i), \\ \gamma_{ni}^*(u) &:= \frac{1}{2h^2} (d^2K)_h(u - Y_{n,i}^*) (Y_{n,i} - Y_i)^{(2)}. \end{aligned}$$

Here  $d^j K$  denotes the  $j$ th derivative of the kernel  $K$  (note that  $K_h(u - Y_{ni}) = \alpha_i(u) + \beta_{ni}(u) + \gamma_{ni}^*(u)$ ). Observing the estimate  $\|Y_{n,i} - Y_i\|_\infty = O_P((\frac{\log_2 n}{n})^{\frac{1}{2}})$  it follows that  $A_{n3} = O_P(\frac{\log_2 n}{nh^2}) = o_P(\frac{1}{\sqrt{n}})$ . For the term  $A_{n2}$  we obtain by a straightforward but tedious calculation (see Appendix A)

$$(3.30) \quad A_{n2} = \frac{1}{n} \sum_{i=1}^n r_n(Y_i) + o_P(\frac{1}{\sqrt{n}}),$$

where we used the notation in (3.23). Finally,  $A_{n4} = O(h^2)$  by a standard calculation, which yields

$$(3.31) \quad \partial_\theta Q_n(\theta_{L^2}^*) - 2 \int (K_h * \tau_0(u) - \tau_0(u)) \partial_\theta \tau(u, \theta_{L^2}^*) \omega(u) du = \frac{2}{n} \sum_{i=1}^n C D_n(Y_i) + o_P(\frac{1}{\sqrt{n}}),$$

where the matrix  $C$  is defined in (3.21). Under the assumptions of Theorem 3.3 it can be shown that the kernel estimate of the copula density converges uniformly in probability to the “true” density  $\tau_0$  on  $[\varepsilon, 1 - \varepsilon]^d$  (for any  $\varepsilon > 0$ ), and it follows

$$(3.32) \quad \begin{aligned} \partial_\theta^2 Q_n(\theta_{L^2}^*) &= 2 \int \left( (\tau_n(u) - \tau(u, \theta_{L^2}^*)) \partial_{\theta\theta}^2 \tau(u, \theta_{L^2}^*) - \partial_\theta \tau(u, \theta_{L^2}^*) \partial_\theta^T \tau(u, \theta_{L^2}^*) \right) \omega(u) du \\ &\xrightarrow{P} -2C. \end{aligned}$$

Observing (3.27) this gives

$$(3.33) \quad \partial_\theta Q_n(\theta_{L^2}^*) = 2(C + o_P(1))(\hat{\theta}_{L^2} - \theta_{L^2}^*).$$

and the assertion (3.18) follows because the matrix  $C$  is positive definite (note that  $\theta_{L^2}^*$  minimizes the function  $Q(\theta)$  and  $\theta_{L^2}^*$  is an interior point of  $\Theta$ ).

Note that the dominating term of the right hand side of (3.18) is a sum of centered i.i.d. random variables and the asymptotic normality now follows from the central limit theorem for triangular arrays and the Cramér-Wold device. For the calculation of the asymptotic covariance we note that

$$(3.34) \quad \begin{aligned} \Sigma_n &= \mathbb{E} [D_n(Y_i) D_n(Y_i)^T] = C^{-1} \mathbb{E} [C D_n(Y_i) (C D_n(Y_i))^T] C^{-1} \\ &= C^{-1} \{E_D^{(1)} + 2E_D^{(2)} + E_D^{(3)}\} C^{-1}, \end{aligned}$$

where

$$\begin{aligned} E_D^{(1)} &= \mathbb{E} \left[ \int (K_h(u - Y_i) - E K_h(u - Y_i)) \partial_\theta \tau(u, \theta_{L^2}^*) \right. \\ &\quad \left. (K_h(v - Y_i) - E K_h(v - Y_i)) \partial_\theta^T \tau(v, \theta_{L^2}^*) \omega(u) \omega(v) du dv \right], \\ 2 E_D^{(2)} &= \mathbb{E} \left[ \int (K_h(u - Y_i) - E K_h(u - Y_i)) \partial_\theta \tau(u, \theta_{L^2}^*) \omega(u) du r_n^T(Y_i) \right], \\ E_D^{(3)} &= \mathbb{E} [r_n(Y_i) r_n^T(Y_i)]. \end{aligned}$$

The assertion of the theorem now follows by a straightforward but tedious evaluation of the expressions  $E_D^{(j)}$  ( $j = 1, 2, 3$ ) observing that

$$r_n(Y_i) = - \sum_{i=1}^d \int (\tau_0 \nabla \tau(\cdot, \theta_{L^2}^*) \omega)(u_1, \dots, u_{r-1}, Y_{ir}, u_{r+1}, \dots, u_d) d\underline{u}_{-r} + O(h)$$

□

**Remark 3.4.** If the condition

$$\|\partial_\theta Q(\theta) |_{\theta=\theta_{L^2}^*}\| < \inf\{\|\partial_\theta Q(\theta)\| \mid \theta \in \Theta; \|\theta - \theta_{L^2}^*\| \geq \varepsilon\}$$

is satisfied for all  $\varepsilon > 0$ , then it has been shown by Bücher (2008) (using Theorem 5.9 in van der Vaart (1998)) that under the bandwidth conditions stated in Theorem 3.3 the estimate  $\hat{\theta}_{L^2}$  is consistent, that is  $\hat{\theta}_{L^2} \xrightarrow{P} \theta_{L^2}^*$ .

## 4 Weak convergence of the statistic $J_n(\hat{\theta})$

In this section we study the asymptotic properties of the goodness-of-fit test for the parametric form of the copula which is based on the  $L^2$ -distance  $J_n$  with the estimator  $\hat{\theta}_{ML}$  or  $\hat{\theta}_{L^2}$ . We begin with a statement of the asymptotic properties of the statistic  $J_n(\hat{\theta})$  under the null hypothesis of a correct specification of the copula family.

**Theorem 4.1.** *If the assumptions stated in Section 2 and 3 are satisfied and*

$$nh^d \rightarrow \infty, \frac{\log_2^2 n}{nh^{4+\frac{d}{2}}} \rightarrow 0$$

, then

$$(4.1) \quad nh^{\frac{d}{2}} \left( J_n(\hat{\theta}_{ML}) - \frac{1}{nh^d} \int K^2(t) \tau_0(u - ht) \omega(u) dt du + \frac{1}{nh} \int \tau_0^2 \omega \sum_{r=1}^d \int K_r^2 \right) \xrightarrow{\mathcal{D}_{\mathcal{H}_0}} \mathcal{N}(0, 2\sigma^2),$$

where the asymptotic variance is given by

$$(4.2) \quad \sigma^2 = \int \tau_0^2 \omega(u) du \int \left( \int K(u) K(u+v) du \right)^2 dv.$$

If additionally  $nh^{4+d} \rightarrow 0$ ,  $nh^{6+\frac{d}{2}} \rightarrow 0$ , then

$$\begin{aligned} nh^{\frac{d}{2}} \left( J_n(\hat{\theta}_{L^2}) - \frac{1}{nh^{\frac{d}{2}}} \int K^2(t) \tau_0(u - ht) \omega(u) dt du + \frac{1}{nh} \int \tau_0^2 \omega \sum_{r=1}^d \int K_r^2 \right. \\ \left. - \int (\partial_\theta^T \tau(u, \theta_0) B)^2 \omega(u) du \right) \xrightarrow{\mathcal{D}_{\mathcal{H}_0}} \mathcal{N}(0, 2\sigma^2), \end{aligned}$$

where  $\sigma^2$  is given as above and

$$(4.3) \quad \begin{aligned} B &= \int (K_h * \tau_0(u) - \tau_0(u)) C^{-1} \partial_\theta \tau(u, \theta_0) \omega(u) du = O(h^2), \\ C &= \int \partial_\theta \tau(u, \theta_0) \partial_\theta^T \tau(u, \theta_0) \omega(u) du. \end{aligned}$$

The proof of this result follows by similar arguments as given by Fermanian (2005), where some modifications are necessary, because the estimators  $\hat{\theta}_{ML}$  and  $\hat{\theta}_{L^2}$  do not satisfy the assumptions (3.1) of this paper. The details are omitted for the sake of brevity.

We now concentrate on the corresponding results under fixed alternatives.

**Theorem 4.2.** *If the assumptions of Theorem 4.1 are satisfied and the null hypothesis is not valid, that is  $\tau_0(\cdot) \neq \tau(\cdot, \theta)$  for all  $\theta \in \Theta$ , then*

$$\sqrt{n} \left( J_n(\hat{\theta}_{ML}) - b_1 \right) \xrightarrow{\mathcal{D}_{\mathcal{H}_1}} \mathcal{N}(0, \sigma_{\mathcal{H}_1}^2).$$

where the bias is given by

$$b_1 = \int (K_h * (\tau_0 - \tau^*))^2(u) \omega(u) du.$$

The asymptotic variance is given by

$$\sigma_{\mathcal{H}_1}^2 = 4 \cdot \{\sigma_{11} + \sigma_{22} + \sigma_{33} + 2\sigma_{12} - 2\sigma_{13} - 2\sigma_{23}\}$$

where

$$\begin{aligned}
\sigma_{11} &:= \text{Var}((\tau_0 - \tau^*) \omega(Y_i)), \\
\sigma_{12} &:= -\mathbb{E} \left[ (\tau_0 - \tau^*)(Y_i) \omega(Y_i) \sum_{r=1}^d \int (\tau_0 - \tau^*) \tau_0 \omega(u_1, \dots, Y_{ir}, \dots, u_d) d\underline{u}_{-r} \right], \\
\sigma_{13} &:= \beta_{ML}^T A^{-1} \mathbb{E} \left[ (\tau_0 - \tau^*) \omega(Y_i) \left( \partial_\theta \log \tau(Y_i, \theta_{ML}^*) \right. \right. \\
&\quad \left. \left. + \partial_{y\theta}^2 \log \tau(Y_j, \theta_{ML}^*) (\mathbb{1}(Y_i \leq Y_j) - Y_j) \right) \right], \\
\sigma_{22} &:= \mathbb{E} \left[ \sum_{r,s=1}^d \int (\tau_0 - \tau^*) \tau_0 \omega(u_1, \dots, Y_{ir}, \dots, u_d) \right. \\
&\quad \left. (\tau_0 - \tau^*) \tau_0 \omega(v_1, \dots, Y_{is}, \dots, v_d) d\underline{u}_{-r} d\underline{v}_{-s} \right], \\
\sigma_{23} &:= \beta_{ML}^T A^{-1} \mathbb{E} \left[ \sum_{r=1}^d \int (\tau_0 - \tau^*) \tau_0 \omega(v_1, \dots, Y_{ir}, \dots, v_d) d\underline{v}_{-r} \right. \\
&\quad \left. \left( \partial_\theta \log \tau(Y_i, \theta_{ML}^*) + \partial_{y\theta}^2 \log \tau(Y_j, \theta_{ML}^*) (\mathbb{1}(Y_i \leq Y_j) - Y_j) \right) \right], \\
\sigma_{33} &:= \beta_{ML}^T \text{Var}(D(Y_i)) \beta_{ML}
\end{aligned}$$

and we have used the notation  $\tau^*(u) = \tau(u, \theta_{ML}^*)$ ,

$$\beta_{ML} = \int (\tau_0 - \tau^*) \omega(u) \partial_\theta \tau(u, \theta_{ML}^*) du.$$

Similarly, we have

$$\sqrt{n} \left( J_n(\hat{\theta}_{L^2}) - b_1 - 2b_2 \right) \xrightarrow{\mathcal{D}_{\mathcal{H}_1}} \mathcal{N}(0, \sigma_{\mathcal{H}_1}^2),$$

where

$$\begin{aligned}
b_1 &= \int (K_h * (\tau_0 - \tau^*))^2(u) \omega(u) du, \\
b_2 &= \int K_h(u-t) (\tau_0 - \tau^*)(t) K_h(u-s) \partial_\theta^T \tau(s, \theta_{L^2}^*) B \omega(u) du ds dt = O(h^3), \\
\sigma_{\mathcal{H}_1}^2 &= 4(\sigma_{11} + \sigma_{22} + 2\sigma_{12}),
\end{aligned}$$

with

$$\begin{aligned}
\sigma_{11} &:= \text{Var}((\tau_0 - \tau^*) \omega(Y_i)), \\
\sigma_{12} &:= -\mathbb{E} \left[ \sum_{r=1}^d \int (\tau_0 - \tau^*) \tau_0 \omega(u_1, \dots, Y_{ir}, \dots, u_d) (\tau_0 - \tau^*) \omega(Y_i) \omega(Y_i) d\underline{u}_{-r} \right], \\
\sigma_{22} &:= \mathbb{E} \left[ \sum_{r,s=1}^d \int (\tau_0 - \tau^*) \tau_0 \omega(u_1, \dots, Y_{ir}, \dots, u_d) \right. \\
&\quad \left. (\tau_0 - \tau^*) \tau_0 \omega(v_1, \dots, Y_{is}, \dots, v_d) d\underline{u}_{-r} d\underline{v}_{-s} \right].
\end{aligned}$$

**Remark 4.3.** Note that the choice  $\omega(\cdot) = \tau(\cdot, \theta_{ML}^*)^{-1}$  yields

$$\begin{aligned}
\beta_{ML} &= \int \tau_0(u) \frac{\partial_\theta \tau(u, \theta_{ML}^*)}{\tau(u, \theta_{ML}^*)} du - \int \partial_\theta \tau(u, \theta_{ML}^*) du \\
&= \int \tau_0(u) \partial_\theta \log \tau(u, \theta) |_{\theta=\theta_{ML}^*} du
\end{aligned}$$

and the term  $\beta_{ML}$  in the first part of Theorem 4.2 vanishes. By a careful inspection of the proof of Theorem 4.2 it can be shown that for the choice  $\omega(\cdot) = \tau(\cdot, \hat{\theta}_{ML})^{-1}$  the asymptotic variance of the statistic  $\sqrt{n}(J_n(\hat{\theta}_{ML}) - b_1)$  simplifies substantially and is given by

$$\sigma_{\mathcal{H}_1}^2 = 4(\sigma_{11} + 2\sigma_{12} + \sigma_{22}).$$

**Proof of Theorem 4.2.** We restrict ourselves to a proof of the first part, the corresponding result for the estimator  $\hat{\theta}_{L^2}$  is derived similarly [see Bücher (2008)]. We use the decomposition

$$(4.4) \quad J_n(\hat{\theta}_{ML}) = W_{n1} + W_{n2} + W_{n3} + W_{n4} + W_{n5} + W_{n6},$$

where the quantities  $W_{nj}$  are given by

$$\begin{aligned}
W_{n1} &= \int (\tau_n - K_h * \tau_0)^2(u) \omega(u) du, \\
W_{n2} &= \int (K_h * (\tau_0 - \tau^*))^2(u) \omega(u) du, \\
W_{n3} &= \int (K_h * (\tau^* - \hat{\tau}))^2(u) \omega(u) du, \\
W_{n4} &= 2 \int (\tau_n - K_h * \tau_0)(u) (K_h * (\tau_0 - \tau^*)) (u) \omega(u) du, \\
W_{n5} &= 2 \int (\tau_n - K_h * \tau_0)(u) (K_h * (\tau^* - \hat{\tau})) (u) \omega(u) du, \\
W_{n6} &= 2 \int (K_h * (\tau_0 - \tau^*)) (u) (K_h * (\tau^* - \hat{\tau})) (u) \omega(u) du,
\end{aligned}$$

$\tau^*(\cdot) = \tau(\cdot, \theta_{ML}^*)$  denotes the best approximation of the “true” copula density  $\tau_0$  by the parametric family with respect to the Kulback Leibler distance and  $\hat{\tau}(\theta) = \tau(\cdot, \hat{\theta}_{ML})$  its corresponding estimate. From the proof of Theorem 4.1 [see Fermanian (2005)] we obtain

$$W_{n1} = o_p(n^{-1/2}), \quad W_{n3} = o_p(n^{-1/2}),$$

and the assertion of the theorem follows, if the weak convergence

$$(4.5) \quad \sqrt{n} (W_{n4} + W_{n5} + W_{n6}) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_{\mathcal{H}_1}^2)$$

can be established. In order to prove this result we investigate the term  $W_{n4}$ ,  $W_{n5}$  and  $W_{n6}$  separately. Recalling the notation in (3.29) we obtain the decomposition

$$W_{n4} = W_{n4}^{(1)} + W_{n4}^{(2)} + W_{n4}^{(3)},$$

where

$$(4.6) \quad W_{n4}^{(1)} = \frac{2}{n} \sum_{i=1}^n Z_i = \frac{2}{n} \sum_{i=1}^n (K_h * g_h)(Y_i) - \mathbb{E}[(K_h * g_h)(Y_i)],$$

$$(4.7) \quad W_{n4}^{(2)} = \frac{2}{n} \sum_{i=1}^n \int \beta_{ni} g_h(u) du,$$

$$(4.8) \quad W_{n4}^{(3)} = \frac{2}{n} \sum_{i=1}^n \int \gamma_{ni}^* g_h(u) du,$$

$g_h(u) = (K_h * (\tau_0 - \tau^*))(u)\omega(u)$  and equation (4.6) defines the random variables  $Z_i$  in an obvious manner. Obviously, the term  $W_{n4}^{(1)}$  is a sum of i.i.d. random variables, and a straightforward but tedious calculation shows that  $W_{n4}^{(3)} = o_p(n^{-1/2})$  (for this estimate we use the conditions on the bandwidth and the estimate  $\|Y_{n,i} - Y_i\|^2 = O_p(\frac{\log_2 n}{n})$ ). For the remaining term we use a further decomposition

$$(4.9) \quad W_{n4}^{(2)} = \frac{2}{n} \sum_{i=1}^n \int \beta_{ni} g_h du = W_{n4}^{(2,1)} + W_{n4}^{(2,2)},$$

where

$$\begin{aligned} W_{n4}^{(2,1)} &= \frac{-2}{n^2 h} \sum_{i=1}^n \int (dK)_h(u - Y_i)(1 - Y_i) g_h(u) du = o_P\left(\frac{1}{\sqrt{n}}\right), \\ W_{n4}^{(2,2)} &= \frac{2}{n^2} \sum_{k \neq i} \frac{-1}{h} \int (dK)_h(u - Y_i)(\mathbb{1}(Y_k \leq Y_i) - Y_i) g_h(u) du. \end{aligned}$$

The second term can be approximated by

$$(4.10) \quad W_{n4}^{(2,2)} = \frac{2}{n} \sum_{i=1}^n s_n(Y_i) + o_P\left(\frac{1}{\sqrt{n}}\right),$$



where

$$(4.11) \quad s_n(Y_i) = \mathbb{E}[k_n(Y_k, Y_i) | Y_i]$$

and

$$k_n(Y_k, Y_i) := \frac{-1}{h} \int (dK)_h(u - Y_k)(\mathbb{1}(Y_i \leq Y_k) - Y_k)g_h(u) du.$$

[see Appendix B.] A standard calculation now shows that

$$s_n(Y_i) = - \sum_{r=1}^d \mathbb{E}[(\tau_0 - \tau^*)\tau_0 \omega(Y_i) | Y_{ir}] + O(h),$$

which gives

$$(4.12) \quad W_{n4} = \frac{2}{n} \sum (Z_i + s_n(Y_i)) + o_p(n^{-1/2}),$$

where  $Z_i$  and  $s_n(Y_i)$  are defined by (4.6) and (4.11), respectively. An application of the Cauchy-Schwarz inequality shows that

$$(4.13) \quad |W_{n5}| \leq 2 \left( W_{n1} W_{n3} \right)^{\frac{1}{2}} = o_P \left( \frac{1}{\sqrt{n}} \right),$$

and for the remaining term  $W_{n6}$  we have

$$(4.14) \quad W_{n6} = -\frac{2}{n} \sum_{i=1}^n B(Y_i) + o_P \left( \frac{1}{\sqrt{n}} \right),$$

where  $B(Y_i) := \beta_{ML}^T D(Y_i)$  as shown in Appendix C. Therefore we obtain from (4.5) that

$$\sqrt{n} (J_n(\hat{\theta}_{ML}) - b_1) = \frac{2}{\sqrt{n}} \sum_{i=1}^n (Z_i + s_n(Y_i) - B(Y_i)) + o_p(1),$$

and the assertion of the theorem now follows from the central limit theorem and a straightforward but tedious calculation of  $\text{Var}(Z_i + s_n(Y_i) - B(Y_i))$ .  $\square$

## 5 Simulation Study

In this section we study the performance of a parametric bootstrap goodness-of-fit test based on the statistic  $J_n(\hat{\theta}_{ML})$ . From Theorem 4.1 we get, using the notation  $\hat{\tau}(\cdot) = \tau(\cdot, \hat{\theta}_{ML})$ , that

$$T_n := nh^{\frac{d}{2}} \frac{J_n - \frac{1}{nh^d} \int K^2(t) \hat{\tau}(u - ht) \omega(u) dt du + \frac{1}{nh} \int \hat{\tau}^2 \omega \sum_{r=1}^d \int K_r^2}{\sqrt{2} \int \hat{\tau}^2 \omega(u) du \int \left( \int K(u) K(u+v) du \right)^2 dv}$$

converges weakly to the standard normal distribution. With  $u_{1-\alpha}$  denoting the  $(1 - \alpha)$ -quantile of the  $\mathcal{N}(0, 1)$ -distribution we therefore obtain, by rejecting  $\mathcal{H}_0$  for  $T_n > u_{1-\alpha}$ , an asymptotic

level- $\alpha$  test. Since this normal approximation does not provide sufficiently exact critical values for small sample sizes [see e.g. Härdle and Mammen (1993) or Fan and Linton (2003)], we propose a parametric bootstrap procedure in order to approximate the critical values. For this purpose we proceed as follows.

In a first step compute the ML-estimate  $\hat{\theta}_{ML}$  and simulate for each  $b = 1, \dots, B$  with  $B \in \mathbb{N}$  independent identically distributed random vectors  $Y_1^{b*}, \dots, Y_n^{b*}$  with distribution function  $C_{\hat{\theta}_{ML}}$ . In a second step we calculate for every of the  $B$  samples the statistic

$$T_n^{b*} = T_n^{b*}(Y_1^{b*}, \dots, Y_n^{b*})$$

and denote by

$$H_{n,b}^*(t) = \frac{1}{B} \sum_{b=1}^B \mathbb{1}\{T_n^{b*} \leq t\}$$

the empirical distribution function of  $T_n^{1*}, \dots, T_n^{B*}$ . We determine the  $(1 - \alpha)$ -quantile of this distribution and use it as a critical value for the goodness-of-fit test statistic  $T_n = T_n(X_1, \dots, X_n)$ .

For our simulation study we use a similar setting as in Genest et. al. (2008). We choose a sample size of  $n = 150$  and check the performance of the parametric bootstrap-procedure for two copula families, namely the Gauss- and the Clayton-Copula. The true copula is chosen from the Clayton-, Gauss-, Frank- or Gumbel-Family with parameter determined by the Kendall's- $\tau$ -coefficient taking values in  $\{0.25, 0.5, 0.75\}$ . We use  $B = 100$  Bootstrap-Replications and make 100 replications of the whole procedure in order to estimate the power of the test. Considering the parameters emerging in the definition of  $T_n$  we choose

$$\begin{aligned} \omega &= \mathbb{1}_{[0.25, 0.975]} \\ h &= 0.7 n^{-1/6} \\ K(u, v) &= (15/16)^2 (1 - u^2)^2 (1 - v^2)^2 \mathbb{1}_{[-1, 1]^2}(u, v) \end{aligned}$$

The results are presented in Table 1. From this table it can be seen that the level of the test is well approximated in almost all cases, but there appears an effect of underestimation for larger  $\tau$ -coefficients.

Regarding the power of the test our results are quite comparable to other simulation studies considering the goodness-of-fit testing for copula families, see e.g. Genest et al. (2008). With stronger dependence, measured by the  $\tau$ -coefficient, the test performs substantially better than in case of weak dependence. In comparison to the tests studied in Genest et al. (2008) it is remarkable that our bootstrap test outperforms all tests within that paper in case of the true copula family being Frank. In the other cases a comparison is more difficult. For example, if the copula under the null hypothesis is Gauss but the true copula is Gumbel, the  $L^2$ -test proposed in this paper yields similar results as most of the tests investigated by Genest et. al. (2008), but there exist also more powerful tests. On the other hand, if the true copula is Clayton, the bootstrap version of

the test proposed by Fermanian (2005) yields a power comparable to the best tests investigated by Genest et. al. (2008). A similar observation can be made if the copula under the null hypothesis is Clayton. In this case the  $L^2$ -test always yields a similar power as the best test considered by Genest. et.al. (2008).

For further conclusions and interpretations of the results, especially in comparison to other tests, we refer to the extensive simulation study in the paper of Genest et al. (2008).

## 6 Appendix

### 6.1 Proof of identity (3.30)

Using the notation

$$(6.1) \quad h_{nm}^*(Y_i, Y_k) := \frac{-1}{h} \int (dK)_h(u - Y_i)(\mathbb{1}(Y_k \leq Y_i) - Y_i) \partial_{\theta_m} \tau(u, \theta_{L^2}^*) \omega(u) du$$

(where  $\partial_{\theta_m}$  denotes the derivative with respect to the  $m$ -th component of the vector  $\theta$ , for  $m = 1, \dots, q$ ) we obtain the decomposition

$$(6.2) \quad A_{n2m} = A_{n2m}^{(1)} + A_{n2m}^{(2)}$$

for the  $m$ -th component of the vector  $A_{n2}$ , where the random variables  $A_{n2m}^{(i)}$  ( $i = 1, 2$ ) are defined by

$$(6.3) \quad A_{n2m}^{(1)} = \frac{-1}{n^2 h} \sum_{i=1}^n \int (dK)_h(u - Y_i)(1 - Y_i) \partial_{\theta_m} \tau(u, \theta_{L^2}^*) \omega(u) du,$$

$$(6.4) \quad A_{n2m}^{(2)} = \frac{1}{n^2} \sum_{k \neq i} h_{nm}^*(Y_i, Y_k).$$

A straightforward standard calculation yields the estimate  $A_{n2m}^{(1)} = o_P(n^{-\frac{1}{2}})$ . The second term  $A_{n2m}^{(2)}$  can be identified as a non-degenerate U-statistic

$$(6.5) \quad A_{n2m}^{(2)} = \frac{1}{n^2} \sum_{i < j} \tilde{h}_{nm} = \frac{n-1}{2n} \binom{n}{2}^{-1} U_n,$$

where  $\tilde{h}_{nm}(Y_i, Y_k) = h_{nm}^*(Y_i, Y_k) + h_{nm}^*(Y_k, Y_i)$  denotes the symmetrized kernel of  $U_n$ . A straightforward but tedious calculation yields the estimate  $\mathbb{E} [\tilde{h}_{nm}^2(Y_i, Y_k)] = O(h^{-2}) = o(n)$ , so that the assumptions of Lemma 3.1 in Zheng (1996) are fulfilled. This result gives

$$(6.6) \quad \tilde{A}_{n2m}^{(2)} - A_{n2m}^{(2)} = o_P(n^{-\frac{1}{2}}),$$

where  $\tilde{A}_{n2m}^{(2)}$  denotes the orthogonal projection  $\tilde{A}_{n2m}^{(2)} = \frac{1}{n} \sum_{i=1}^n \tilde{r}_{nm}(Y_i)$  and  $\tilde{r}_{nm}(Y_i) := \mathbb{E} [\tilde{h}_{nm}(Y_i, Y_k) | Y_i]$ . Finally, observing that  $\mathbb{E} [h_{nm}^*(Y_i, Y_k) | Y_i = y_i] = 0$ , it follows that  $r_{nm}(Y_i) = \tilde{r}_{nm}(Y_i)$  which yields the assertion in equation (3.30).  $\square$

Copula under $\mathcal{H}_0$	True Copula	$\tau$ -Coeff	0.15	0.1	0.05
Clayton	Clayton	0.25	0.14	0.1	0.08
		0.5	0.12	0.08	0.03
		0.75	0.08	0.05	0.01
	Frank	0.25	0.81	0.74	0.63
		0.5	1	1	0.98
		0.75	1	1	1
	Gauss	0.25	0.63	0.54	0.41
		0.5	0.96	0.95	0.91
		0.75	1	1	0.98
	Gumbel	0.25	0.89	0.83	0.73
		0.5	0.99	0.99	0.98
		0.75	1	1	1
Gauss	Clayton	0.25	0.4	0.31	0.21
		0.5	0.89	0.84	0.77
		0.75	0.99	0.99	0.97
	Frank	0.25	0.5	0.4	0.25
		0.5	0.9	0.83	0.73
		0.75	1	1	1
	Gauss	0.25	0.15	0.07	0.02
		0.5	0.12	0.1	0.04
		0.75	0.15	0.04	0.01
	Gumbel	0.25	0.2	0.18	0.14
		0.5	0.41	0.31	0.17
		0.75	0.37	0.26	0.16

Table 1: *Simulated rejection probabilities of the  $L^2$ -type test for various null hypotheses and alternatives. The sample size is  $n = 150$  and  $B = 100$  parametric Bootstrap-replications have been performed. The last three columns show the percentage of rejection of  $\mathcal{H}_0$  at level  $\alpha \in \{0.15, 0.1, 0.05\}$ .*

## 6.2 Proof of identity (4.10)

The proof of equation (4.10) follows along the same lines as the one given in appendix A via application of Lemma 3.1 in Zheng (1996). Using the notation  $\tilde{k}_n(Y_i, Y_k) := k_n^*(Y_i, Y_k) + k_n^*(Y_k, Y_i)$  for the symmetrized kernel we obtain the following identification of  $W_{n4}^{(2,2)}$  as a non-degenerate U-statistic:

$$(6.7) \quad W_{n4}^{(2,2)} = \frac{2}{n^2} \sum_{i < k} \tilde{k}_n(Y_i, Y_k) = \frac{n-1}{n} \binom{n}{2}^{-1} U_n.$$

A straightforward calculation shows  $\mathbb{E}[\tilde{k}_n^2(Y_i, Y_k)] = O(h^{-2}) = o(n)$ , and an application of Lemma 3.1 in Zheng (1996) yields

$$(6.8) \quad \widehat{W}_{n4}^{(2,2)} - W_{n4}^{(2,2)} = o_P(n^{-\frac{1}{2}}),$$

where  $\widehat{W}_{n4}^{(2,2)}$  denotes the orthogonal projection  $\widehat{W}_{n4}^{(2,2)} = \frac{2}{n} \sum_{i=1}^n \tilde{s}_n(Y_i)$  with  $\tilde{s}_n(Y_i) = \mathbb{E}[\tilde{k}_n(Y_i, Y_k) | Y_i]$ . Observing that  $\mathbb{E}[k_n^*(Y_i, Y_k) | Y_i = y_i] = 0$  we obtain  $\tilde{s}_n(Y_i) = s_n(Y_i)$  and the assertion follows.  $\square$

## 6.3 Proof of identity (4.14)

By means of a Taylor expansion we obtain the decomposition

$$(6.9) \quad W_{n6} = W_{n6}^{(1)} + W_{n6}^{(2)},$$

where the random variables  $W_{n6}^{(i)}$  are defined by

$$\begin{aligned} W_{n6}^{(1)} &= -2 \int K_h(u-t)(\tau_0 - \tau^*)(t) K_h(u-s) \partial_\theta^T \tau(s, \theta_{ML}^*) (\widehat{\theta}_{ML} - \theta_{ML}^*) \omega(u) dt ds du, \\ W_{n6}^{(2)} &= - \int K_h(u-t)(\tau_0 - \tau^*)(t) K_h(u-s) (\widehat{\theta}_{ML} - \theta_{ML}^*)^T \partial_{\theta\theta}^2 \tau(s, \tilde{\theta}) \\ &\quad (\widehat{\theta}_{ML} - \theta_{ML}^*) \omega(u) dt ds du, \end{aligned}$$

for some  $\tilde{\theta}$  with  $\|\tilde{\theta} - \theta_{ML}^*\| \leq \|\widehat{\theta}_{ML} - \theta_{ML}^*\|$ . A straightforward calculation yields

$$W_{n6}^{(2)} = O_P \left( \left\| \widehat{\theta}_{ML} - \theta_{ML}^* \right\|^2 \right) = O_P(n^{-1}) = o_P(n^{-\frac{1}{2}}).$$

Using identity (3.4) from Theorem 3.1 we obtain

$$W_{n6}^{(1)} = -\frac{2}{n} \sum_{i=1}^n \int K_h(u-t)(\tau_0 - \tau^*)(t) K_h(u-s) \partial_\theta^T \tau(s, \theta_{ML}^*) \omega(u) du dt ds D(Y_i) + o_P \left( \frac{1}{\sqrt{n}} \right).$$

Finally, considering expansions of  $(\tau_0 - \tau^*)$  and  $\omega$ , the dominating sum can be estimated by  $\frac{-2}{n} \sum_{i=1}^n \beta_{ML}^T D(Y_i) + o_P(n^{-\frac{1}{2}})$ , which yields the assertion in equation (4.14).  $\square$

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